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## Ductile, Brittle Failure Characteristics as Determined by the State of the Material and the Imposed State of Stress

Richard M. Christensen

### Abstract

A method is developed for determining whether a particular mode of failure is expected to be of ductile type or brittle type depending upon both the state of the material and the particular state of stressing the isotropic material to failure. The state of the material is determined by two specific failure properties and a newly formulated failure theory. The ductile versus brittle criterion then involves the state of the material specification and the mean normal stress part of the imposed stress state. Several examples are given for different stress states and a spectrum of materials types. Closely related to the failure mode types are the orientations of the associated failure surfaces. The resulting failure surface angle predictions are compared with those from the Coulomb-Mohr failure criterion. In uniaxial tension, only the present method correctly predicts the octahedral failure angle at the ductile limit, and also shows a distinct failure mode transition from ductile type to brittle type as the state of the material changes. The explicit D-B criterion and the related failure surface orientation methodology are intended to provide a refinement and generalization of the ductile-brittle transition viewed only as a state property to also include a dependence upon the type of stress state taken to failure.

## Introduction and Background

For some purposes it is not sufficient to simply view a particular material as being either ductile or brittle in any and all possible conditions. Behaviors of these types also depend upon the particular state of stress under which the material is taken to failure. For example, superimposed pressure can convert what is commonly thought of as a brittle material into a ductile material for a particular stress state taken to failure. The total stress state is a decisive factor. A stress state based failure criterion that distinguishes fracture from plastic flow could provide a step toward a more discriminating approach to the ductile versus brittle failure judgement.

Failure criteria for materials are widely employed but not particularly well understood. There has been a long history going back to the incipient interest in material science as a scholarly discipline. The sub-field of materials failure however has largely developed by empirical means. There are only two classical forms, the two parameter Coulomb-Mohr form for general applications and the one parameter Mises form for ductile metals. The Tresca form can be considered to be a special case of the Coulomb-Mohr form. Three or more parameter forms generally have evolved simply as a means of having flexibility in fitting data. If the two parameter Coulomb-Mohr form had been successful in evaluations with a variety of materials types, the quest would have been closed long ago. However, it has been unsuccessful in that regard. Nevertheless the advantages of a two property failure form are apparent and the present work is concerned with a completely different type of two parameter (property) theory of failure for isotropic materials, as an alternative to both the two parameter Coulomb-Mohr form and the many multi-parameter empirical forms.

This recently developed failure theory has been given by Christensen [1], also giving references to earlier work and a rather complete historical account for the field. The treatment admits the Mises form for ductile materials at one extreme and a brittle, limiting case behavior at the other extreme. A specific criterion also is given for the associated ductile to brittle (D-B) transition. This criterion provides the means for using testing results as input to determine whether the general state of the material is ductile or brittle. Obvious examples for each class abound. The ductile versus brittle state of a material at specified conditions of pressure and temperature is a global property which

depends upon the physical and chemical composition of the material and all other sources that contribute to and define the state of the material. However, in contrast to any such means of specifying the general D-B state for a given material, particular states of imposed stress may show quite different failure mode characteristics. Each failure mode often can be observed as being of ductile or brittle nature, but not necessarily in harmony with the general D-B state of the material. For example tensile stress states are far more likely to show brittle failure characteristics than are compressive stress states, regardless of the D-B state of the material. The present work is entirely concerned with developing a method for determining which type of failure mode (D or B) is likely to occur based upon both the state of the material and the particular state of stressing the material to failure. Neither specification by itself is sufficient to determine the expected D-B nature of a specific failure mode. The state of the material will be quantified through the new failure criteria mentioned earlier.

This new criterion for homogeneous, isotropic material failure, Christensen [1], was derived in the tensor form given below

$$\alpha \left( \frac{\sigma_{ii}}{\kappa} \right) + \frac{3}{2} (1 + \alpha) \left( \frac{s_{ij}}{\kappa} \right) \left( \frac{s_{ij}}{\kappa} \right) \leq 1 \quad (1)$$

and

$$\sigma_P \leq \frac{\kappa}{1 + \alpha} \quad \text{if } \alpha > 1 \quad (2)$$

where

$$\begin{aligned} \kappa &= |\sigma_{11}^C| = C \\ \alpha &= \frac{|\sigma_{11}^C|}{\sigma_{11}^T} - 1 \\ &= \frac{C}{T} - 1 \quad (0 \leq \alpha \leq \infty) \end{aligned} \quad (3)$$

and where  $s_{ij}$  is the deviatoric stress tensor, and  $\sigma_P$  is the largest principal stress. Thus, the failure theory is fully characterized by the uniaxial tensile and compressive stress values T & C. Criterion (2) is a fracture cutoff effect, which places further limits on (1).

Examples will be given later. This criterion functions both as a yield or failure criterion depending upon the application. The ductile-brittle transition was found to be given by

$$\begin{aligned}\alpha < 1 & \quad \text{Ductile} \\ \alpha = 1 & \quad \text{Transition} \\ \alpha > 1 & \quad \text{Brittle}\end{aligned}\tag{4}$$

The details of the derivation leading to (1)-(4) are given in the reference previously cited. The criteria (4) will be taken as specifying the general state of the material, according to the material specific value for  $\alpha$ .

It is important to recognize the significance of the fracture criterion (2). Although maximum principle stress has often been used as a fracture criterion, it is also necessarily specified in (2) that it only applies in the brittle range of behavior, given by  $\alpha > 1$ , according to (4). This latter restriction is of fundamental importance, and provides necessary coordination with the other physical effects implicit in the yield/strength criterion (1).

It also is important to place the ductile-brittle (D-B) criteria (4) in the proper context. Usually the D-B transition is considered in the context of dislocations emissions from crack tips. Whether or not a dislocation flow can cause shielding or blunting then is used to relate to the D-B criterion. For a prime example of this well motivated approach, see Rice and Thomson [2], as well as other works. In the present work the interests are with much broader classes of materials than just those controlled by dislocations. For this reason the macroscopic D-B criteria (4) are employed here.

In complete component form (1), (2) and (4) become

$$\begin{aligned}& \left(\frac{1}{T} - \frac{1}{C}\right)(\sigma_{11} + \sigma_{22} + \sigma_{33}) \\ & + \frac{1}{TC} \left\{ \frac{1}{2} \left[ (\sigma_{11} - \sigma_{22})^2 + (\sigma_{22} - \sigma_{33})^2 + (\sigma_{33} - \sigma_{11})^2 \right] \right. \\ & \left. + 3(\sigma_{12}^2 + \sigma_{23}^2 + \sigma_{31}^2) \right\} \leq 1\end{aligned}\tag{5}$$

$$\sigma_P \leq T \text{ if } \frac{T}{C} \leq \frac{1}{2} \quad (6)$$

and

$$\begin{aligned} \frac{1}{2} < \frac{T}{C} &\leq 1 && \text{Ductile} \\ \frac{T}{C} &= \frac{1}{2} && \text{Transition} \\ 0 &\leq \frac{T}{C} < \frac{1}{2} && \text{Brittle} \end{aligned} \quad (7)$$

In cases where the tensile and shear failure stress,  $S$ , are known but not the compressive failure stress, then from (5) it follows that the compressive value is given by

$$C = \frac{3S^2}{T}$$

Some examples of materials in the ductile range according to criteria (7) and at ambient conditions are ductile metals and some polymers, particularly thermoplastics. Examples in the brittle range are cast iron, ceramics, glasses and minerals. Glassy polymers are typically near the transition. The failure criteria (5) and (6) were extensively and favorably compared with data from many of these material types, Christensen [1]. Specifically, rather complete failure data sets for polypropylene, iron, a graphitic material and dolomite were compared with predictions from (5) – (7) and found to be much better than those from Coulomb-Mohr, and generally satisfactory.

It should be mentioned that the material classes intended to be covered by this criteria are those for  $0 \leq \alpha \leq \infty$  which includes virtually all homogeneous, isotropic materials. Material types which are excluded are those with a significant amount of porosity which typically have  $T/C > 1$  giving negative  $\alpha$ . It is now seen that the fracture cutoff criterion (2) or (6) only applies for a material type which is brittle according to the criterion (7).

In principle stress space the criterion (5) represents a paraboloid, inclined at equal angles to all three principle axes and opening up in the negative coordinate directions. The fracture cut off criterion (6) involves three planes normal to the principle stress axes which then intersect the paraboloid in cases of brittle materials per criterion (7). This rather intricate and non-intuitive failure surface still devolves from only two failure

parameters. In contrast, the two parameter Coulomb-Mohr form is simply a six sided pyramid in principal stress space, Paul [2]. Even more simple is the conical failure surface proposed by Drucker and Prager [3], also involving two parameters. With the two parameters adjusted to uniaxial tensile and compressive strengths, both the Coulomb-Mohr and Drucker-Prager forms strongly overestimate strengths in the tensile region of their apexes as well as in the opposite very compressive regions of their failure surfaces, Paul [2]. In a state of equal tri-axial tension, the Coulomb-Mohr method predicts a failure stress three times larger than that from (1)-(3), for all values of  $\alpha$ . Some three parameter failure forms may be seen in Paul [2] and in Jaeger and Cook [4]. The specific paraboloid form (5) has had some discussion and a few applications, see for example Raghava, Caddell and Yeh [5]. The present complete set of criteria (1)-(6) are of recent origin and have no historical precedent. The ductile-brittle failure characteristics (4), (7) were first discussed in the materials context by Christensen [6, 7].

#### A Ductile versus Brittle Criterion Determined by the Imposed State of Failure Stress and the State of the Material

It will be helpful to rewrite the failure criteria as a hybrid combination of the tensor form (1) and (2) and the component form (5) and (6). Let stress be non-dimensionalized as

$$\hat{\sigma}_{ij} = \frac{\sigma_{ij}}{\kappa} = \frac{\sigma_{ij}}{C} \quad (8)$$

Then the failure criteria are

$$\begin{aligned} & \alpha \left( \hat{\sigma}_{11} + \hat{\sigma}_{22} + \hat{\sigma}_{33} \right) \\ & + (1 + \alpha) \left\{ \frac{1}{2} \left[ \left( \hat{\sigma}_{11} - \hat{\sigma}_{22} \right)^2 + \left( \hat{\sigma}_{22} - \hat{\sigma}_{33} \right)^2 + \left( \hat{\sigma}_{33} - \hat{\sigma}_{11} \right)^2 \right] \right. \\ & \left. + 3 \left( \hat{\sigma}_{12}^2 + \hat{\sigma}_{23}^2 + \hat{\sigma}_{31}^2 \right) \right\} \leq 1 \end{aligned} \quad (9)$$

and

$$\hat{\sigma}_p \leq \frac{1}{1+\alpha} \quad \text{if } \alpha > 1 \quad (10)$$

The advantage of this form is that while the stresses are specified in terms of components, the non-dimensional failure property  $\alpha$ , (3), can be used to scan different material types as  $\alpha$  varies from 0 to  $\infty$ . In effect, property  $\alpha$  will be used to represent the state of the material in accordance with (4). The objective now is to determine whether a given mode of failure is ductile or brittle according to the specification of the state of the material given by  $\alpha$  in (4) and the specified homogeneous state of stress of interest.

Begin with the two-dimensional biaxial stress state involving  $\sigma_{11}$  and  $\sigma_{22}$  with all other stress components as vanishing. The criterion (9) becomes

$$\alpha \left( \hat{\sigma}_{11} + \hat{\sigma}_{22} \right) + (1+\alpha) \left( \hat{\sigma}_{11}^2 - \hat{\sigma}_{11} \hat{\sigma}_{22} + \hat{\sigma}_{22}^2 \right) \leq 1 \quad (11)$$

The fracture cutoff criterion is still given by (10), with  $\sigma_p$  as the largest principle stress.

The biaxial failure criteria (10) and (11) give the failure forms shown in Fig. 1 for various values of  $\alpha$ . These four figures are illustrative and drawn to approximate scale. The fracture cutoff behavior first begins to intercede at  $\alpha = 1$ , becomes quite prominent at  $\alpha = 2$  and becomes very dominant at large values of  $\alpha$ . Cast iron is a good material example at  $\alpha = 2$  and ceramics are materials with large values of  $\alpha$ , Christensen [1]. The case  $\alpha = 0$  is that of a very ductile metal controlled by the Mises criterion.

The boundary between ductile and brittle regions of behavior is taken as the dotted lines shown in Fig. 1c and 1d. Thus this boundary is at the edge of the fracture cutoff zones, as shown. The brittle zone is toward the tension-tension quadrant in Fig. 1 while the ductile zone is toward the compression-compression quadrant. The brittle zone is controlled by two separate brittle fracture mechanisms. The fracture cutoff form shown in Fig. 1 is due to the usual crack opening mode of fracture involving a single stress component. The remaining part of the brittle zone in the first quadrant is of a more involved interaction effect between the stress components. In Fig. 1b, the fracture cutoff zones have shrunk to two points. It will be useful to evaluate the position of the fracture cutoff zone in terms of parameter  $\alpha$ . This position coordinate is given by coordinate  $\zeta$  as shown in Figs. 1b and 1d. The yield/failure function is given by (11) and the fracture form by (10). Take (10) as

$$\hat{\sigma}_{22} = \frac{1}{1+\alpha} \quad (12)$$

Substitute (12) into (11) to find

$$\hat{\sigma}_{11} = \frac{1-\alpha}{1+\alpha} \quad (13)$$

Relations (12) and (13) give the location of the intersection of the yield/failure surface and the fracture cutoff in the second quadrant. The similar point in the fourth quadrant is given by

$$\left. \begin{aligned} \hat{\sigma}_{11} &= \frac{1}{1+\alpha} \\ \hat{\sigma}_{22} &= \frac{1-\alpha}{1+\alpha} \end{aligned} \right\} \quad (14)$$

The mid point of the dashed lines in Fig. 1 can be found from (12) – (14) and thence the position coordinate  $\zeta$  as

$$\zeta = \frac{2-\alpha}{\sqrt{2}(1+\alpha)} \quad (15)$$

The line defined by  $\zeta$ , (15), shown in Fig. 1 separates the ductile and brittle regions. Assuming that this expression for  $\zeta$  in the range  $\alpha > 1$  also smoothly continues in the range  $\alpha < 1$  then this gives the tangent condition at  $\alpha = 0$  shown in Fig. 1a.

Next, define any two points on the failure/yield surface (11) by another line parallel to the line given by  $\zeta$  in Fig. 1. Designate its position coordinate by  $\eta$ . From the basic geometry it follows that

$$\eta = \frac{1}{\sqrt{2}} \left( \hat{\sigma}_{11} + \hat{\sigma}_{22} \right) \quad (16)$$

where  $\hat{\sigma}_{11}$  and  $\hat{\sigma}_{22}$  must satisfy (11). This line designated by coordinate  $\eta$ , parallel to that designated by  $\zeta$  in Fig. 1 must be to the left of the dotted lines in Fig. 1 for ductile behavior to exist. This condition is then stated as

$$\eta < \zeta \quad , \quad \text{Ductile} \quad (17)$$

Combining (15)-(17) gives the 2-D criterion as

$$\hat{\sigma}_{11} + \hat{\sigma}_{22} < \frac{2-\alpha}{1+\alpha} \quad , \text{ Ductile, 2-D} \quad (18)$$

where  $\hat{\sigma}_{11}$  and  $\hat{\sigma}_{22}$  must satisfy (11) for the specified stress state. This then is the ductile–brittle criterion for a particular stress state under 2-D conditions. Before discussing this criterion it is best to obtain the corresponding generalization of it.

Now consider three dimensional stress states. The previous 2-D case can be viewed as states in 3-D principle stress space  $\hat{\sigma}_{11}$ ,  $\hat{\sigma}_{22}$  and  $\hat{\sigma}_{31}$  with one of the principle stresses as vanishing. In the 2-D case the D-B criterion (18) is seen to have the stress state specified in terms of its mean normal stress, or the first stress invariant. This feature is also evident in Fig. 1. For a particular state of the material the boundary between ductile and brittle behavior is specified by a particular value of the mean normal stress for the applied stress state. This same characteristic will be generalized to any 3-D stress state. As discussed in the Introduction Section the failure surface (9) in principal stress space is a paraboloid of revolution intersected by three planes normal to the principle stress axes, representing the fracture cutoff effect (10). The mean normal stress (first stress invariant) defines a plane normal to the line that makes equal angles to axes  $\sigma_1$ ,  $\sigma_2$ , and  $\sigma_3$ . This plane intersects the failure surface and is taken to separate it into two regions, one of ductile failure and one of brittle failure. The location of this 3-D boundary plane is taken such that it intersects the coordinate planes  $\sigma_1$ ,  $\sigma_2$ , and  $\sigma_2$ ,  $\sigma_3$  and  $\sigma_3$ ,  $\sigma_1$  according to the criteria already established in the 2-D stress state case. This unambiguously defines the location of the mean normal stress specified boundary between the ductile and brittle regions of the 3-D failure surface.

Carrying out this 3-D procedure, the criterion for the determination of the failure mode type is found to be given by the direct generalization of (18) as

$$\begin{aligned} \hat{\sigma}_{11} + \hat{\sigma}_{22} + \hat{\sigma}_{33} &< \frac{2-\alpha}{1+\alpha} \quad , \text{ Ductile} \\ \hat{\sigma}_{11} + \hat{\sigma}_{22} + \hat{\sigma}_{33} &> \frac{2-\alpha}{1+\alpha} \quad , \text{ Brittle} \end{aligned} \quad (19)$$

where the stress state of interest involved in the left hand side of (19) must satisfy the failure criterion (9). The first stress invariant (three times the mean normal stress) on the left side of (19) is specified by the applied stress state and the right side of (19) designates the state of the material, through  $\alpha$ . This then is the ductile-brittle criterion as determined by the stress state and the state of the material. The criterion in (19) is invariant to the dimensionality, being of the same form for 1-D, 2-D and 3-D cases.

In addition to the D-B criterion (19) related to the yield-failure criterion (9), it also must be remembered that the fracture cutoff regions themselves are of inherently brittle behavior. The overall brittle region then is of the front part of the paraboloid failure surface specified by (19) along with the three fracture cutoff surfaces that penetrate the ductile region somewhat further, giving a type of three lobed surface in that region.

The criterion (19) for the failure mode type involves the mean normal stress part of the applied stress state. This is consistent with the overall failure criteria (9) and (10). If in (9) there is no mean normal stress effect, then that would require  $\alpha = 0$  and the resulting behavior is completely ductile, as governed by the Mises criterion. There must be a mean normal stress effect in this failure criterion for brittle behavior to ensue. Now some examples will be given to illustrate the method and reveal specific results.

The uniaxial stress case with  $\hat{\sigma}_{11} \neq 0$  and all other components as vanishing leaves (19) as

$$\hat{\sigma}_{11} < \frac{2-\alpha}{1+\alpha} \quad , \quad \text{Ductile} \quad (20)$$

For tensile stress, the solution from (9) is

$$\hat{\sigma}_{11}^T = \frac{1}{1+\alpha} \quad (21)$$

Combining (21) into (19) then gives

$$\alpha < 1 \quad , \quad \text{Ductile} \quad (22)$$

Thus uniaxial tension is of ductile failure if  $\alpha < 1$  and it is brittle if  $\alpha > 1$ .

For uniaxial compressive stress, the solution from (9) is

$$\hat{\sigma}_{11}^C = -1 \quad (23)$$

which combined with (20) gives

$$-1 < \frac{2-\alpha}{1+\alpha} \quad , \quad \text{Ductile} \quad (24)$$

which is always satisfied except as  $\alpha \rightarrow \infty$ . Thus uniaxial compression is always of ductile failure. It is not surprising that uniaxial compressive failure is always of the ductile type even though tension is not. The uniaxial compressive state has a mean normal stress component that is itself of hydrostatic compression and promotes ductile behavior.

A state of shear stress is given by

$$\hat{\sigma}_{11} = -\hat{\sigma}_{22} \quad (25)$$

Inserting (25) into criterion (19) gives

$$\alpha < 2 \quad , \quad \text{Ductile} \quad (26)$$

Thus shear stress produces ductile failure if the state of the material is  $\alpha < 2$  but it produces brittle failure if  $\alpha > 2$ .

Next consider a typical 3-D stress state. Take

$$\hat{\sigma}_{11} = \hat{\sigma}_{22} = \hat{\sigma}_{33} = \hat{\sigma}$$

and

$$\begin{aligned} \hat{\sigma}_{12} &= 0 \\ \hat{\sigma}_{23} &= \left| \hat{\sigma} \right| \\ \hat{\sigma}_{31} &= 0 \end{aligned} \quad (27)$$

This is a shear stress with a superposed hydrostatic stress, of the same magnitude. The D-B criterion (19) becomes

$$3\hat{\sigma} < \frac{2-\alpha}{1+\alpha} \quad , \quad \text{Ductile} \quad (28)$$

Inserting stress state (27) into the failure criterion (9) gives

$$\hat{\sigma} = \frac{\alpha}{2(1+\alpha)} \left[ -1 \pm \frac{\sqrt{3\alpha^2 + 4\alpha + 4}}{\sqrt{3}\alpha} \right] \quad (29)$$

for the tensile and compressive roots of  $\hat{\sigma}$ . Combining (28) and (29) in the case of tensile  $\hat{\sigma}$  gives

$$\alpha < \frac{1}{2}, \quad \text{Ductile} \quad (30)$$

as the ductile range. The complementary range,  $\alpha > \frac{1}{2}$ , will be brittle behavior.

Combining (28) and (29) in the case of compressive  $\hat{\sigma}$  gives

$$0 \leq \alpha < \infty, \quad \text{Ductile} \quad (31)$$

Thus this mode of failure is always ductile. This case with  $\hat{\sigma}$  compressive shows an example of how a material with  $\alpha > 2$ , which is brittle in simple shear stress, when placed under superimposed pressure (27) is converted into a ductile material in this stress state.

Now several specific cases will be summarized, based upon the criteria (9) and (19) and as illustrated in the previous examples. The D-B criteria will be specified in terms of T/C values since uniaxial tensile and compressive failure values in this ratio are readily recognizable for different materials.

General State of Material

$$\frac{1}{2} < \frac{T}{C} \leq 1 : \text{Ductile}$$

$$0 < \frac{T}{C} < \frac{1}{2} : \text{Brittle}$$

Uniaxial Tension

$$\frac{1}{2} < \frac{T}{C} \leq 1 : \text{Ductile}$$

$$0 < \frac{T}{C} < \frac{1}{2} : \text{Brittle}$$

Uniaxial Compression

$$\text{All } \frac{T}{C} : \text{Ductile}$$

$$\frac{T}{C} \rightarrow 0 : \text{Borderline}$$

Eqi-Biaxial Tension

$$0 < \frac{T}{C} < 1 \quad : \text{ Brittle}$$

$$\frac{T}{C} = 1 \quad : \text{ Borderline}$$

Eqi-Biaxial Compression

$$\text{All } \frac{T}{C} \quad : \text{ Ductile}$$

Eqi-Triaxial Tension

$$\text{All } \frac{T}{C} \quad : \text{ Brittle}$$

Eqi-Triaxial Compression

No Yielding or Failure

Simple Shear

$$\frac{1}{3} < \frac{T}{C} \leq 1 \quad : \text{ Ductile}$$

$$0 < \frac{T}{C} < \frac{1}{3} \quad : \text{ Brittle}$$

For these various states of stress, only that of uniaxial tension is in accordance with that of the general state of the material. For all other cases the failure mode type may or may not be synchronized with the D-B state of the material, depending upon the value of the state variable,  $\alpha$ . For any value of  $\alpha$ , that is for a given material, this methodology can be used to determine the expected type of mode of failure for any given stress state. This is of considerable interest since the usual sudden nature of brittle type failures generally have more severe consequences than those ductile type failures.

It is interesting to consider the case of the failure mode type shown above for a simple shear state of stress, this being  $\alpha < 2$  for ductile failure. In the case of uniaxial tension  $\alpha < 1$  gives ductile failure. Comparison of these two cases then shows that the class of ductile materials in shear is larger than the class of ductile materials in uniaxial tension. Thus there is a class of materials in the intermediate range,  $1 < \alpha < 2$ ,  $\frac{1}{3} < \frac{T}{C} < \frac{1}{2}$ , that would be ductile in shear but brittle in uniaxial tension.

The case of uniaxial compression is especially interesting. The results here show this failure mode to be of ductile type for all materials in this study class. However, in the extreme limit  $T/C \rightarrow 0$ , representing ultimately damaged materials, then the failure type is borderline between ductile and brittle. There have been research studies which characterize uniaxial compressive failure as being some variation of brittle fracture. The present approach suggests that brittle fracture in uniaxial compression is not possible, except in the limiting case. However, examples with the present methodology also show that materials that are normally thought of as being brittle, such as ceramics (with  $\alpha$  about 10) under a stress state of major unidirectional compression, but with a slight transverse tensile stress do cross over to brittle failure. Thus predominately but not totally compressive stress states can cause brittle failure. It is necessary to examine each particular stress state to ascertain its type of failure; intuition can be a misleading guide

The important case of equi-biaxial tension will be discussed later. Finally, some qualifications and cautions should be noted. This methodology is intended to provide guidance upon a classically difficult problem. It is not really expected that a very small change in a state variable would instantly switch a failure mode type from ductile to brittle, or the reverse. Rather, the approach here is prompted by the fundamental importance of ductile-brittle transitions even if as usually treated in an idealized manner.

### Failure Surface Orientation

The orientation of a failure surface for a material loaded to failure is intimately related to the D-B state of the material. Materials that are normally considered to be brittle typically fail at  $90^\circ$  relative to the load direction for uniaxial tension. A failure angle analysis will now be developed with the intention to examine its relationship to ductile-brittle behavior generally, and specifically with respect to results in the preceding section.

The onset of failure in the present context represents the termination of the reversible elastic state of deformation and the onset of irreversible deformation. The associated flow rule of plasticity is taken to describe the irreversible components of the deformation, as

$$\dot{\epsilon}_{ij}^p = \lambda \frac{\partial f}{\partial \sigma_{ij}} \quad (31)$$

where  $\dot{\epsilon}_{ij}^p$  is the increment of inviscid plastic strain, and  $f()$  is the yield/failure function shown in (9) with  $\lambda$  as a proportionality factor. It is well established that the flow form (31) does not apply for granular materials, such as soils. It is, however, taken to apply here for application to materials which have strong degree of physical coherence and structural integrity established through atomic bonds. This includes materials ranging in type from ideally ductile metals through brittle ceramics.

The orientation of the failure surfaces will now be found for states of uniaxial tension and compression. Using the yield/failure function (9), with nondimensional stress in the flow rule (31) with  $\hat{\sigma}_{11}$  being the uniaxial stress then gives for uniaxial tension

$$\begin{aligned} \dot{\epsilon}_{11}^p &= \lambda(2 + \alpha) \\ \dot{\epsilon}_{22}^p &= \dot{\epsilon}_{33}^p = \lambda(-1 + \alpha) \end{aligned} \quad (32)$$

and for uniaxial compression

$$\begin{aligned} \dot{\epsilon}_{11}^p &= -\lambda(2 + \alpha) \\ \dot{\epsilon}_{22}^p &= \dot{\epsilon}_{33}^p = \lambda(1 + 2\alpha) \end{aligned} \quad (33)$$

where  $\lambda$  is now nondimensional.

Axis 1 is in the direction of the applied stress, with axes 2 and 3 normal to it. Now take a rotated coordinate system  $1'$ ,  $2'$ , and  $3'$  such that axis  $1'$  lies in the plane of the failure surface, whatever orientation that may be. Take this as a rotation about axis 3 such that axis  $2'$  is now normal to the failure surface. Let angle  $\beta$  be the angle of the coordinate rotation thus  $\beta$  is the angle between the applied stress direction and the failure surface. Using the tensor transformations, then in the tensile case relations (32) give the rotated components as

$$\begin{aligned}
\frac{\left(\begin{smallmatrix} \bullet p \\ \varepsilon_{11} \end{smallmatrix}\right)'}{\lambda} &= (2 + \alpha) \cos^2 \beta - (1 - \alpha) \sin^2 \beta \\
\frac{\left(\begin{smallmatrix} \bullet p \\ \varepsilon_{22} \end{smallmatrix}\right)'}{\lambda} &= (2 + \alpha) \sin^2 \beta - (1 - \alpha) \cos^2 \beta \\
\frac{\left(\begin{smallmatrix} \bullet p \\ \varepsilon_{12} \end{smallmatrix}\right)'}{\lambda} &= 3 \sin \beta \cos \beta
\end{aligned} \tag{34}$$

In order to determine the failure angle,  $\beta$ , take the increment of plastic strain in the plane of the failure surface as non-active and vanishing, while the other two increments in (34) are allowed to grow and flow in an unlimited manner implying failure. Thus take

$$\left(\begin{smallmatrix} \bullet p \\ \varepsilon_{11} \end{smallmatrix}\right)^{\odot} = 0 \tag{35}$$

which with (34) gives

$$\tan \beta = \sqrt{\frac{2 + \alpha}{1 - \alpha}} \quad , \quad \alpha \leq 1 \tag{36}$$

It is seen that (36) gives  $\beta = 90^\circ$  at  $\alpha = 1$ , reaching the limit. For values of  $\alpha > 1$ , the fracture cutoff effect (10) intercedes and brittle fracture with the same procedure using (31) gives

$$\beta = 90^\circ \quad , \quad \alpha \geq 1 \tag{37}$$

In the uniaxial compressive case, the same tensor transformation procedure, but using (33), gives the plasticity components as

$$\begin{aligned}
\frac{\left(\begin{smallmatrix} \bullet p \\ \varepsilon_{11} \end{smallmatrix}\right)'}{\lambda} &= -(2 + \alpha)\cos^2 \beta + (1 - 2\alpha)\sin^2 \beta \\
\frac{\left(\begin{smallmatrix} \bullet p \\ \varepsilon_{22} \end{smallmatrix}\right)'}{\lambda} &= -(2 + \alpha)\sin^2 \beta + (1 - \alpha)\cos^2 \beta \\
\frac{\left(\begin{smallmatrix} \bullet p \\ \varepsilon_{12} \end{smallmatrix}\right)'}{\lambda} &= -3(1 + \alpha)\sin \beta \cos \beta
\end{aligned} \tag{38}$$

Again setting the increment of irreversible strain  $\left(\begin{smallmatrix} \bullet p \\ \varepsilon_{11} \end{smallmatrix}\right)^{\odot}$  equal to zero such that this

component in the failure plane cannot grow in an unlimited manner the way the other two out of plane components grow in failure, gives

$$\tan \beta = \sqrt{\frac{2 + \alpha}{1 + 2\alpha}} \tag{39}$$

The results of this failure angle analysis are shown in Tables 1 and 2. The results correctly predict that in the ductile limit,  $\alpha = 0$ , the failure angle is the octahedral angle  $\beta = \tan^{-1} \sqrt{2} = 54.7^\circ$ , Nadai [8]. The  $\alpha \rightarrow \infty$  limits are given by  $90^\circ$  in the case of tension and  $\beta = \tan^{-1} (1 / \sqrt{2}) = 35.3^\circ$  in the case of compression.

It is interesting to compare the failure angle behavior just found with that predicted by the Coulomb-Mohr method. From Paul [2] the Coulomb-Mohr failure angles are given by

$$\beta = 45^\circ + \frac{\phi}{2} \quad , \quad \text{Tension}$$

and

$$\beta = 45^\circ - \frac{\phi}{2} \quad , \quad \text{Compression}$$

(40)

where

$$\phi = \sin^{-1} \left( \frac{\frac{C}{T} - 1}{\frac{C}{T} + 1} \right)$$

In the ductile limit  $C=T$ , the Coulomb-Mohr result gives  $\beta = 45^\circ$  and does not even correctly predict the octahedral angle.

The present failure angle predictions and those from Coulomb-Mohr are shown in Fig. 2. It is seen that they produce fundamentally different failure surface orientations. Only in the limit,  $\alpha \rightarrow \infty$  does the Coulomb-Mohr method correctly predict a fracture angle of  $\beta = 90^\circ$  in tension. Typical glassy, brittle polymers such as polystyrene, PMMA, and some epoxies can have uniaxial tension to compression ratios as low as about 1/2. Experimental results show these materials to fail brittly in tension with a failure angle of  $\beta = 90^\circ$ , Berry [9], see also Ward [10]. Coulomb-Mohr predicts this failure angle from (40) as  $54.7^\circ$ . The present method (36), (37) correctly predicts  $\beta = 90^\circ$ . The Coulomb-Mohr method also predicts  $\beta \rightarrow 0$  as  $\alpha \rightarrow \infty$  in compression whereas the present method predicts the limiting angle as  $\beta = 35.3^\circ$ . Most importantly, the Coulomb-Mohr method completely misses a failure angle manifestation of the ductile-brittle transition at  $\alpha = 1$ , whereas the present method shows a strong change in failure mode type in tension at  $\alpha = 1$ , Fig. 2. In uniaxial compression, Fig. 2, the present failure angle behavior is consistent with that of a single type, and thereby consistent with the previously found result that this mode of failure is always ductile.

### Conclusions

The present work is comprised of a method for determining whether a particular mode of failure can be expected to be of a ductile or a brittle nature based upon the state of the homogeneous material and the imposed state of stress. Numerous examples are shown where the nominal state of the material can be either of ductile or brittle type, but the mode of failure for a particular stress state will be of the opposite type. Probably the most important example of this behavior is that for a tensile equi-biaxial stress state, which is always of brittle failure type even though a particular material may be thought of as being nominally ductile. Thin shell pressure vessels provide an example where such situations can arise and likely warrant special consideration because of the implied fracture danger.

Perhaps the most interesting aspect of these results is the comparison of the behaviors under states of uniaxial tension and compression, certainly two of the most basic states. In uniaxial compression the failure mode was found to always be of ductile type, regardless of the state of the material, but in uniaxial tension the failure mode type can be of either ductile or brittle type depending upon the state of the material. Specific criteria were given for locating this D-B transition in uniaxial tension.

The compressive strength being ductile does not mean that failure occurs without the appearance of cracks. As shown in the previous section, even in compression there is an identifiable failure plane which may even lead to crack formation as the localized damage state intensifies. Nevertheless, ductile failure modes would be expected to be considerably different and less sudden or intensive than brittle ones.

The Coulomb-Mohr failure criterion was shown to give unrealistic predictions for failure surface orientations. Furthermore, it shows no evidence of a specific transition from ductile behavior to brittle behavior. This is not too surprising when one considers its derivation from frictional type behavior in granular materials. It essentially is a failure form motivated by and most likely applicable to highly damaged materials. Conversely, for the present interests in materials that show a high degree of physical integrity, namely metals, polymers, and ceramics, the associated flow rule (31) was found to provide a consistent and useful tool for failure analysis.

Yielding and fracture are competitive physical effects. Although various materials types are often viewed as being either brittle or ductile, in an all inclusive manner. That characterization is in some cases overly simplistic and any given material can exhibit either brittle or ductile failure mechanisms depending upon the state of stress. That is the main result of this work. In the details of the present work, the boundary or border separating the ductile region from the brittle region of stress space was given by the intersection of the yield criterion and the fracture criterion. None of the present results could have been found were it not for this failure specification involving both the yield form (1) and the fracture criterion based upon maximum principal stress (2). The two criteria are intimately and even essentially interrelated, and control complementary regions of behavior.

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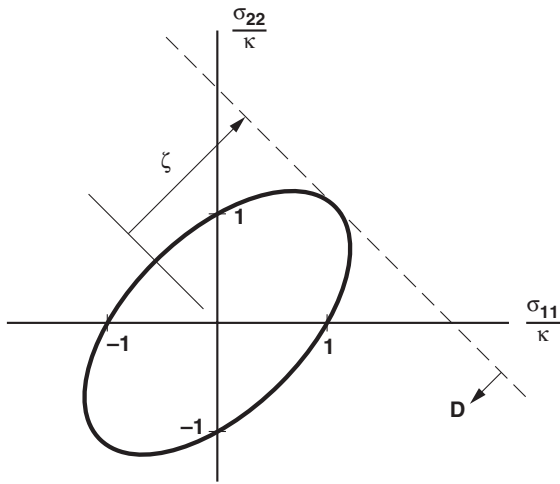
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Material	$\alpha$	$\frac{T}{C}$	Tensile $\beta$ Degrees
Ductile Metals	0	1	54.7
Ductile Metals & Polymers	$\frac{1}{4}$	$\frac{4}{5}$	60
Ductile Polymers	$\frac{1}{2}$	$\frac{2}{3}$	65.9
Transition, Glassy Polymers	1	$\frac{1}{2}$	90
Brittle Materials	$>1$	$<\frac{1}{2}$	90

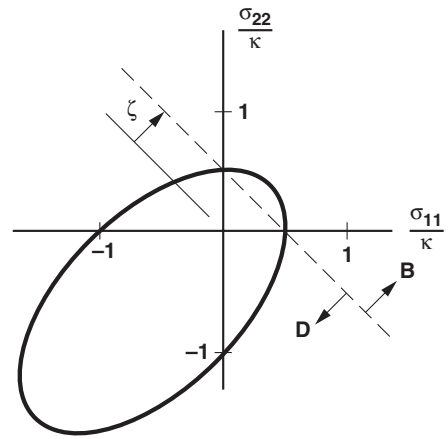
Table 1. Failure Angles, Uniaxial Tension, Eqs. (36) and (37)

Material	$\alpha$	$\frac{T}{C}$	Compressive $\beta$ Degrees
Ductile Metals	0	1	54.7
Ductile Polymers	$\frac{1}{2}$	$\frac{2}{3}$	48.2
Transition, Glassy Polymers	1	$\frac{1}{2}$	45
Cast Iron	2	$\frac{1}{3}$	41.8
Ceramics	10	$\frac{1}{11}$	37.1
Brittle Limit	$\rightarrow\infty$	$\rightarrow 0$	35.3

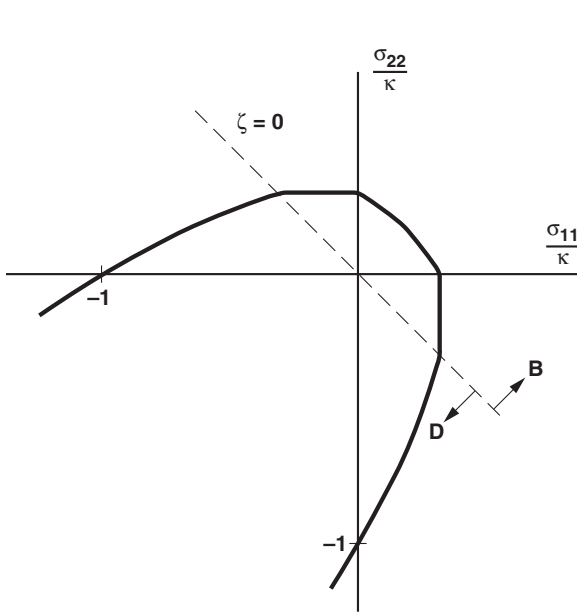
Table 2. Failure Angles, Uniaxial Compression, Eq. (39)



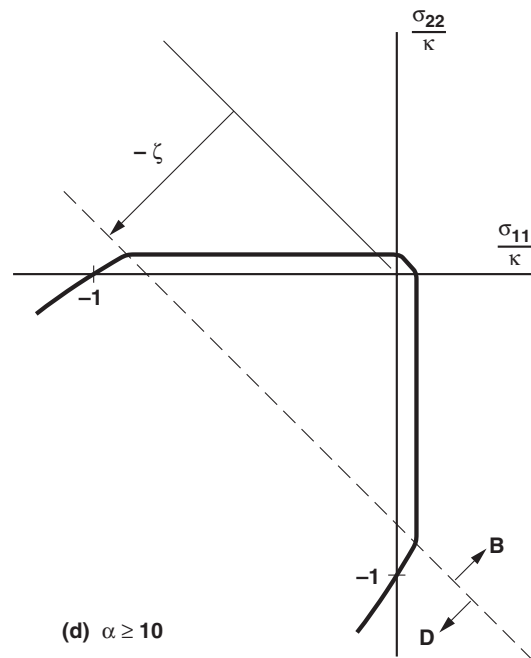
(a)  $\alpha = 0$



(b)  $\alpha = 1$



(c)  $\alpha = 2$



(d)  $\alpha \geq 10$

Figure 1. Bi-Axial Stress States

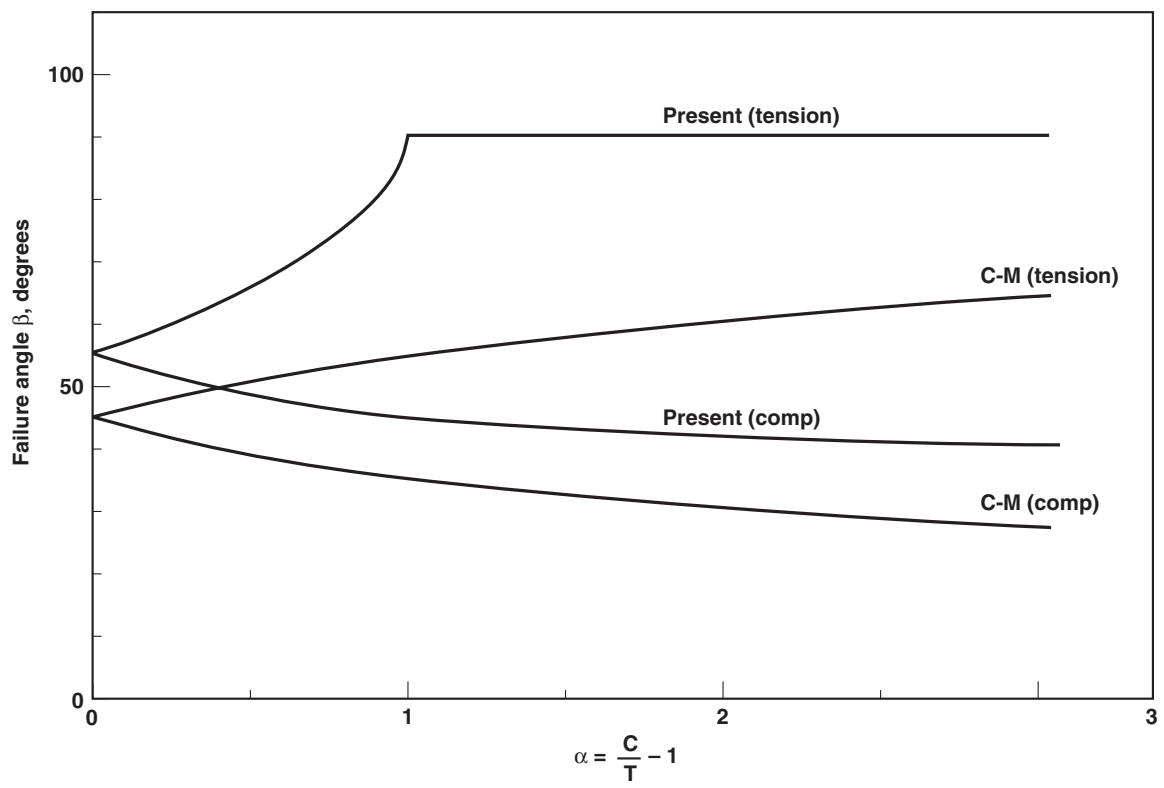


Figure 2. Failure Angle Predictions